1) $u(x,y) = xy - x + y$.

a. u is harmonisch op $\mathbb{R}^2$ als

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = y - 1 \qquad \frac{\partial u}{\partial y} = x + 1$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0.$$

Dus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0$

$\rightarrow$ u is harmonisch op $\mathbb{R}^2$

b. Cauchy-Riemann equations: (C-R equations)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$f(z) = u(x,y) + iv(x,y)$

$f(z)$ is analytic on $\mathbb{C}$ if:

i) the first partial deratives of u and v exist.

ii) the first partial deratives are continuous.

iii) the Cauchy-Riemann equations hold.

~~Als of~~ If $f(z)$ is analytic, then it is harmonic on $\mathbb{R}^2$.

So I'm going to find an harmonic conjugate $v(x,y)$ of $u(x,y)$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = y - 1 \; (*) \quad , \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -x - 1.$$

Integrate (*) with respect to y:

$$v(x,y) = \int (y-1)\,dy = \tfrac{1}{2}y^2 - y + \psi(x)$$

$$\frac{\partial v}{\partial x} = \psi'(x) = -x - 1 \quad \rightarrow \psi(x) = -\tfrac{1}{2}x^2 - x$$

So ~~the~~ a harmonic conjugate $v(x,y)$ of $u(x,y)$ is

$v(x) = \tfrac{1}{2}y^2 - y - \tfrac{1}{2}x^2 - x$

(which is a real function, since the imaginary part is zero)

So $f(z) = u(x,y) + iv(x,y)$

$= xy - x + y + i\left(\frac{1}{2}y^2 - y - \frac{1}{2}x^2 - x\right)$

~~check~~ just to check:

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= y - 1 \\ \frac{\partial v}{\partial y} &= y - 1\end{aligned}\right\} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\left.\begin{aligned}\frac{\partial u}{\partial y} &= x + 1 \\ \frac{\partial v}{\partial x} &= -x - 1\end{aligned}\right\} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i) The first partial derivatives exist

ii) They are continuous

iii) The C-R equations hold for all $z \in \mathbb{C}$.

So $f(z)$ is analytic on $\mathbb{C}$.

C. v is harmonic if $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

$$\frac{\partial v}{\partial x} = -x - 1 \qquad \frac{\partial v}{\partial y} = y - 1$$

$$\frac{\partial^2 v}{\partial x^2} = -1 \qquad \frac{\partial^2 v}{\partial y^2} = 1$$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -1 + 1 = 0$, so

$v(x,y)$ is harmonic on $\mathbb{R}^2$.

2) $g(z) = \dfrac{\cos z}{z(z-\pi)}$

which has singularities at $z=0$ and $z=\pi$

~~a.~~

a. $\Gamma_1 := |z|=3$, doorlopen in positieve richting.

9

The only singularity within the contour Γ_1, is $z=0$

Define $f(z) = \dfrac{\cos(z)}{z-\pi}$, which is analytic inside and on the contour Γ_1.

Cauchy Integral formula: $\int_{\Gamma_1} \dfrac{f(z)}{z-z_0}\,dz = 2\pi i \cdot f(z_0)$

$$\oint_{\Gamma_1} g(z)dz = \int_{\Gamma} \frac{\cos(z)/(z-\pi)}{z}\,dz = 2\pi i \cdot f(0) = 2\pi i \cdot \frac{\cos(0)}{0-\pi}$$

$$= \frac{2\pi i \cdot 1}{-\pi} = -2i$$

The Residue theorem gives the same result;

$$\oint_{\Gamma_1} g(z)dz = 2\pi i \cdot \mathrm{Res}(g(z);0)$$

$$\mathrm{Res}(g(z);0) = \lim_{z\to 0} z \cdot g(z) = \lim_{z\to 0} \frac{\cos z}{z-\pi} = \frac{\cos(0)}{0-\pi} = \frac{1}{-\pi}$$

$$\oint_{\Gamma_1} g(z)dz = 2\pi i \cdot \frac{1}{-\pi} = -2i$$

b. $\Gamma_2 := |z|=4$, doorlopen in positieve richting.

9

$z=0$ and $z=\pi$ both lie inside the contour.

By the Residue theorem:

$$\int_{\Gamma_2} g(z)dz = 2\pi i\,[\mathrm{Res}(0) + \mathrm{Res}(\pi)].$$

$\mathrm{Res}(g(z);0) = \dfrac{1}{-\pi}$ (see a)

$$\mathrm{Res}(g(z);\pi) = \lim_{z\to\pi} (z-\pi)\cdot g(z) = \lim_{z\to\pi} (z-\pi)\cdot \frac{\cos z}{z(z-\pi)}$$

$$= \lim_{z\to\pi} \frac{\cos(z)}{z} = \frac{\cos(\pi)}{\pi} = \frac{-1}{\pi}$$

$$\int_{\Gamma_2} g(z)dz = 2\pi i\,[\mathrm{Res}(0) + \mathrm{Res}(\pi)] = 2\pi i\left(\frac{-1}{\pi} - \frac{1}{\pi}\right) = 2\pi i \cdot \frac{-2}{\pi}$$

$$= -4i$$

3) $$f(z) = \frac{z(z-\pi)^2}{(\sin z)^2}$$

a. z_0 is een singulariteit van f als

$$\sin z_0 = 0 \longleftrightarrow z_0 = k\pi \quad , k \in \mathbb{Z}.$$

3/3 2

~~[illegible]~~

~~[illegible]~~

b. $z_0 = \pi$ is a removable singularity. $(k=1)$

$$\lim_{z\to\pi} f(z) = \lim_{z\to\pi} \frac{z(z-\pi)^2}{(\sin z)^2} \overset{L'H}{=} \lim_{z\to\pi} \frac{(z-\pi)^2 + 2z(z-\pi)}{2\sin(z)\cos(z)}$$

$$\overset{L'H}{=} \lim_{z\to\pi} \frac{2(z-\pi)+2(z-\pi)+2z}{2\cos^2(z) - 2\sin^2(z)}$$

5/5

$$= \frac{2(\pi-\pi)+2(\pi-\pi)+2\pi}{2\cos^2(\pi) - 2\sin^2(\pi)}$$

$$= \frac{0+0+2\pi}{2\cdot(-1)^2 - 2\cdot 0} = \frac{2\pi}{2} = \pi$$

So by ~~redefining~~ Redefining $f(z)$ you can make $f(z)$ analytic at $z=\pi \rightarrow z=\pi$ is a removable singularity.

c. $f(z)$ has a pole of order m at $z=z_0$ if it is possible to define $f(z)$ as

$$f(z) = (z-z_0)^m g(z), \quad \text{where } g(z) \text{ is analytic at } z_0 \text{ and } g(z_0) \neq 0.$$

$$g(z) = \frac{f(z)}{z} = \frac{(z-\pi)^2}{\sin(z)}$$

~~$f'(z) = \frac{(\sin(z))^2 \cdot ((z-\pi)^2 + 2z(z-\pi)) - z(z-\pi)^2 \cdot 2\sin(z)\cos(z)}{(\sin z)^4}$~~

vervolg (30)

~~aan $z=k\pi$:~~ ~~waa~~ at $z_0=k\pi$

$$\sin(z) = \sin(k\pi) + \cos(k\pi)(z-k\pi) - \frac{1}{2!}\sin(k\pi)(z-k\pi)^2$$
$$- \frac{1}{3!}\cos(k\pi)(z-k\pi)^3 + \frac{1}{4!}\sin(k\pi)(z-k\pi)^4$$
$$+ \frac{1}{5!}\cos(k\pi)(z-k\pi)^5 - \frac{1}{6!}\sin(k\pi)(z-k\pi)^6 + \dots$$

For ~~$k\pi$ = odd kis an odd~~

k = odd:

~~sin sin~~ $\sin(z) = -(z-k\pi) + \frac{1}{3!}(z-k\pi)^3 - \frac{1}{5!}(z-k\pi)^5 + \frac{1}{7!}(z-k\pi)^7 - \dots$

For k = even:

$$\sin(z) = (z-k\pi) - \frac{1}{3!}(z-k\pi)^3 + \frac{1}{5!}(z-k\pi)^5 - \frac{1}{7!}(z-k\pi)^7 + \dots$$

$(\sin z)^2$ For k = odd and k = even.

$$(\sin(z))^2 = (z-k\pi)^2 - \frac{2}{3!}(z-k\pi)^3 + \frac{2}{5!}(z-k\pi)^5 - \frac{1}{3!\,5!}(z-k\pi)^8$$

~~For~~ ~~(vermenigvuldig gewoon alle termen van $\sin(z)$ met~~

$$= \left[-(z-k\pi) + \frac{1}{3!}(z-k\pi)^3 - \frac{1}{5!}(z-k\pi)^5 + \dots\right] \cdot$$
$$\left[-(z-k\pi) + \frac{1}{3!}(z-k\pi)^3 - \frac{1}{5!}(z-k\pi)^5 + \dots\right]$$

(Ik ga niet term voor term berekenen, want die zijn toch triviaal)

$$f(z) = \frac{z(z-\pi)^2}{(\sin z)^2} = \frac{z(z-\pi)^2}{(z-k\pi)^2 - \frac{2}{3!}(z-k\pi)^3 + \frac{2}{5!}(z-k\pi)^5 - \dots}$$

Je kunt hier ~~uit~~ zien dat als $k=1$, $f(z) = \dfrac{z(z-\pi)^2}{(z-\pi)^2 - \frac{2}{3!}(z-k\pi)^3 + \dots}$

$$= \frac{z}{1 - \frac{2}{3}(z-k\pi)^3 + \dots}$$ ← (rest van termen zijn triviaal

Voor $k=1$: $f(\pi) = \dfrac{\pi}{1-0+0\dots} = \pi$

dus ik heb nogmaals bewezen dat $z_0=\pi$ een "Removable singularity" is ~~on~~ =-)

!!

3) c $\quad z_0 = k\pi$, for $k=0$, $z_0 = 0$

$$g(z) = z \cdot f(z) = \frac{z^2 \frac{2}{z}(z-\pi)^2}{(z-k\pi)^2 \pm \frac{2}{3!}(z-k\pi)^3 + \frac{2}{5!}(z-k\pi)^5 - \dots}$$

?

$z_0 = 0$ en $k = 0$, dus

$$g(z) = \frac{z(z-\pi)^2}{z^2 - \frac{2}{3!}z^3 + \frac{2}{5!}z^5 - \dots}$$

0/5

$$= \frac{(z-\pi)^2}{1 - \frac{2}{3!}z^3 + \frac{2}{5!}z^5 - \dots}$$

$$g(0) = \frac{\pi^2}{1} = \pi^2 \neq 0$$

and $g(z)$ is analytic at $z_0 = 0$.

So $z_0 = 0$ is also a pole of order 2.

Ok, van de kandidaten $z_0 = k\pi$, met $k \in \mathbb{Z}$, heb ik geen pool van orde 1 gevonden, alleen maar polen van orde 2, dus er is geen pool met orde 1.

Anders blijft de term onder de deelstreep van $f(z)$ namelijk naar 0 gaan als $z \to k\pi$.

3) d.

If $f(z)$ has a pole of order m at $z_0 = k\pi$, we can write $f(z)$ as

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

For $z_0 = k\pi$, not $k \neq 1$, $k \neq 0$, so $k = \pm 2, \pm 3, \pm 4, \ldots$

$$g(z) = (z-k\pi)^2 \cdot f(z) = \frac{z(z-\pi)^2 \cdot (z-k\pi)^2}{(z-k\pi)^2 - \frac{2}{3!}(z-k\pi)^3 + \frac{2}{5!}(z-k\pi)^5 - \ldots}$$

$$= \frac{z(z-\pi)^2}{1 - \frac{2}{3!}(z-k\pi) + \frac{2}{5!}(z-k\pi)^3 - \ldots}$$

$$g(k\pi) = \frac{k\pi(k\pi-\pi)^2}{1-0+0-\ldots} = k\pi(k\pi-\pi)^2 \neq 0$$

So $g(z)$ is analytic at $z_0 = k\pi$ with $k \neq 1$, and $g(z_0) = g(k\pi) \neq 0$

~~Furthermore, we can write f(z) as~~

$z_0 = k\pi$ with $k \neq 1$, $k \neq 0$, so $k = \pm 2, \pm 3, \pm 4, \ldots$ is a pole of $f(z)$ with order 2.

4) $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$

(2) a. z_0 is a singularity if $z_0^2+1=0$.
$z_0^2 = -1 \rightarrow z_0 = \pm i$

(4) b. $\text{Res}(f;+i) = \lim_{z\to i} (z-i)\cdot f(z) = \lim_{z\to i} (z-i)\cdot\frac{1}{(z-i)(z+i)}$
$= \lim_{z\to i} \frac{1}{z+i} = \frac{1}{i+i} = \frac{1}{2i} = -\frac{1}{2}i$

$\text{Res}(f;-i) = \lim_{z\to -i} (z+i)\cdot f(z) = \lim_{z\to -i} (z+i)\cdot\frac{1}{(z-i)(z+i)}$
$= \lim_{z\to -i}\frac{1}{z-i} = \frac{1}{-i-i} = \frac{1}{-2i} = \frac{1}{2}i$

~~c. Stel $\rho > 1$. Toon aan dat $|f(z)| < \frac{1}{\rho^2-1}$ voor $|z|=\rho$~~

~~Let C be the contour $|z|=\rho$~~

~~$\left|\int_C f(z)dz\right| \leq \max_{z \text{ on } C} |f(z)|\cdot L(C)$~~
~~where $L(C)$ is the length of contour C;~~
~~$L(C) = 2\pi\rho$~~

~~$\rho > 1$, so the singularities $z=i$ and $z=-i$ lie within the contour $|z|=\rho$.~~
~~According to the residue theorem:~~
~~$\int_C f(z)dz = 2\pi i\left[\text{Res}(f;i) + \text{Res}(f;-i)\right]$~~
~~$= 2\pi i\left(-\frac{1}{2}i + \frac{1}{2}i\right) = 2\pi i\cdot 0 = 0$~~
~~$\max_{z \text{ on } C}|f(z)| \geq \frac{1}{L(C)}\cdot\left|\int_C f(z)dz\right| = \frac{1}{2\pi\rho}\cdot 0 = 0.$~~

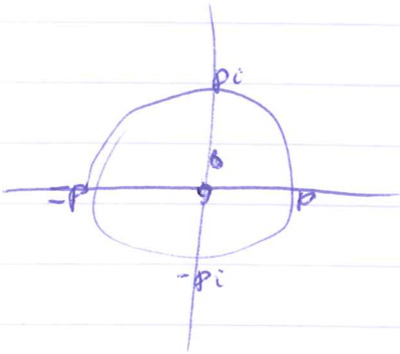
wordt vervolgd

4) c ~~According to the maximum modulus theore~~

4

$|z|=\rho$ is a closed and bounded, so $|f(z)|$ has a maximum on the domain $|z|\leq\rho$.

According to the maximum modulus theorem, the maximum of $|f(z)|$ lies on the boundary of the domain, which is $|z|=\rho$.



~~We say that $\max_{|z|=\rho} |f(z)|$~~

$\rho > 1$, so $\dfrac{1}{\rho^2-1}$ is always a positive term.

$$|f(z)| = \left|\frac{1}{z^2+1}\right| = \frac{1}{|z^2+1|}.$$

Triangle inequality:

$$|z^2| = |z^2+1-1| \leq |z^2+1| + |-1| = |z^2+1|+1$$

$$|z^2+1| \geq |z|^2-1$$

$$\text{So } |f(z)| = \frac{1}{|z^2+1|} \leq \frac{1}{|z|^2-1} = \frac{1}{\rho^2-1}$$

4) d $C_\rho^+ = \{ z \mid Im(z) \geq 0 \text{ en } |z| = \rho \}$.

For C_ρ^+: $z(t) = \rho e^{it}$, with $0 \leq t \leq \pi$

$z'(t) = i\rho e^{it}$.

$$\cancel{\int_{C_\rho^+} \frac{1}{z^2+1} dz =}$$

$$\int f(z)dz = \int f(z(t)) z'(t) dt$$

$$\oint_{C_\rho^+} f(z) dz = \oint_{C_\rho^+} \frac{1}{z^2+1} dz = \int_0^\pi \frac{1}{(\rho e^{it})^2+1} \cdot i\rho e^{it} dt$$

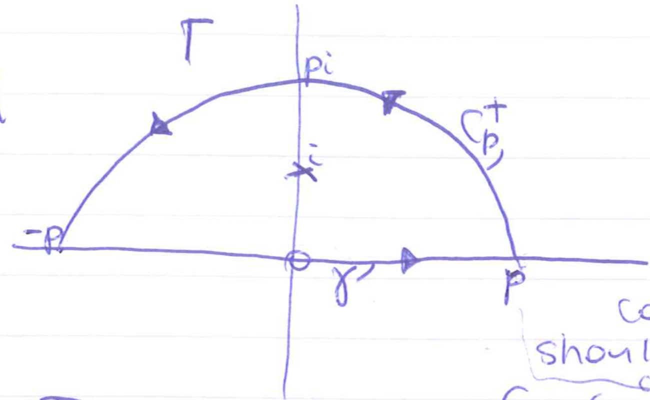
$$= \int_0^\pi \frac{i\rho e^{it}}{\rho^2 e^{2it}+1} dt = \int_0^\pi \frac{ie^{it}}{\rho e^{2it}+1} dt.$$

$$\lim_{\rho \to \infty} \int_{C_\rho^+} f(z) dz = \lim_{\rho \to \infty} \int_0^\pi \frac{ie^{it}}{\rho e^{2it}+1} dt = \int_0^\pi \lim_{\rho \to \infty} \frac{ie^{it}}{\rho e^{2it}+1} dt.$$

$$= \int_0^\pi 0 \, dt = 0$$

4) e

3



$C_p^+ = \{ z \mid Im(z) \geq 0 \text{ en } |z| = p \}$

γ is negatively oriented, so when computing the integral a minus sign should be placed before the integral, to compensate for the fact that it is not positively oriented.

Toon aan dat $\lim_{p\to\infty} \int_{C_p^+} f(z)\,dz = 0$.

$f(z)$ has singularities at $z=+i$ and $z=-i$, but only $z=+i$ lies within the contour defined by $\Gamma = C_p^+ + \gamma$.

$$\text{p.v.} \int_{-\infty}^{\infty} f(z)\,dz = \lim_{p\to\infty} \int_{-p}^{p} \frac{1}{z^2+1}\,dz$$

$$\int_\Gamma \frac{1}{z^2+1}\,dz = 2\pi i \cdot \underset{\text{(see b)}}{Res}(i) = 2\pi i \cdot -\tfrac{1}{2} i = -\pi$$

$$\lim_{p\to\infty} \int_\Gamma \frac{1}{z^2+1}\,dz = -\lim_{p\to\infty} \int_{-p}^{p} \frac{1}{z^2+1}\,dz + \lim_{p\to\infty} \int_{C_p^+} \frac{1}{z^2+1}\,dz$$

$\lim_{p\to\infty} \int_{C_p^+} \frac{1}{z^2+1} = 0$, ~~since the degree of the polynomial~~ since the degree of the denominator is $\geq$ 2+ degree numerator. (and see 4d)

$$\int_{-\infty}^{\infty} f(z)\,dz = \lim_{p\to\infty} \int_{-p}^{p} \frac{1}{z^2+1}\,dz = -\lim_{p\to\infty} \int_\Gamma \frac{1}{z^2+1}\,dz + 0$$

$$= +\pi - 0 = \pi$$

So $\int_{-\infty}^{\infty} f(z)\,dz = \pi$

5) Stelling van Rouché:

6/6

Let C be a simple closed contour and $f(z)$ and $h(z)$ are analytic inside and on the contour C.
If the strict inequality

$|h(z)| < |f(z)|$ holds for all z on the contour C,

then $f(z)$ and $f(z) + h(z)$ have the same amount of zeros (counting multiplicities) inside C.

b. ~~g(z)~~ $= p(z) = z^5 + 4z - 2.$

C is the simple closed contour (circle) $|z| < 2$.

6/6

Define:

$f(z) = z^5$

$h(z) = 4z - 2.$

$f(z)$ and $h(z)$ are analytic inside and on the contour C,

- For z on the contour C, i.e. $|z| = 2$:

$|f(z)| = |z^5| = 32.$

$|h(z)| = |4z - 2| \leq |4z| + |-2| = 4|z| + 2 = 4 \cdot 2 + 2 = 10.$

So $|h(z)| < |f(z)|$ for z on the contour C.
Therefore, $f(z)$ and $p(z) = f(z) + h(z) = z^5 + 4z - 2$ have the same amount of zeros inside C.

A polynomial of degree n, has n zeros, so $f(z) = z^5$ has 5 zeros, (and $p(z)$ also has 5 zeros in total)
These zeros of $f(z)$ all lie inside C, since the zeros of $f(z)$ are $z = 0$ with multiplicity 5.
So $p(z) = z^5 + 4z - 2$ also has 5 zeros inside the ~~contour~~ contour C. (all its zeros lie in the area $|z| < 2$.)

5) c. Ik neem aan dat je bij opgave 5c $p(z)$ bedoelt als je $g(z)$ schrijft.

4/6

~~Define~~ Define C_2 as the simple closed contour $|z| < \frac{1}{4}$.

~~Let $f(z) = 4z - 2$. $f(z)$ has an zero at $z = \frac{1}{2}$, so $f(z)$ has no zeros inside the contour C_2.~~

~~$h(z) = z^5 - 2$~~ ~~$f(z) = 4z$~~ ~~$z^5 - 2$~~

~~$p(z) = f(z) + h(z) = z^5 + 4z - 2$.~~ ~~$h(z) = 4z$~~

(For ~~z on the contour C_2 i.e. $|z| = \frac{1}{4}$~~)

$\boxed{f(z) \text{ and } h(z) \text{ are analytic inside and on the contour } C.}$

~~$|f(z)| = |4z - 2| \geq |4||z| - 2 = \ldots = -1$~~

Triangle inequality:

~~$|4z| = |4z - 2 + 2| \leq |4z - 2| + 2 = |4z - 2| + 2$.~~

~~So $|4z - 2| \geq |4z| - 2$~~

~~Now For z on $|z| = \frac{1}{4}$~~

~~$|f(z)| = |z^5 - 2| \leq |z^5| + 2 = \left(\frac{1}{4}\right)^5 + 2 =$~~

~~$f(z) = 4z - 2$. $f(z)$ has zero at $z = \frac{1}{2}$, which so $f(z)$ has no zeros inside the contour C_2.~~

~~$h(z) = z^5$. $p(z) = h(z) + f(z) = z^5 + 4z - 2$.~~

~~For z on C_2:~~

~~$|h(z)| = |z^5| = \left(\frac{1}{4}\right)^5$~~

~~$|f(z)| = |4z - 2|$~~

~~$f(z)$ has~~ $z^5 - 2 = 0 \to z^5 = 2$ (−2)

$z = \cancel{\sqrt[5]{2}} > \frac{1}{4}$,

So all zeros of $f(z)$ lie outside the contour $|z| = \frac{1}{4}$.

$f(z) = z^5 - 2$

$h(z) = 4z$

For z on $|z| = \frac{1}{4}$:

$|f(z)| = |z^5 - 2| \leq |z^5| + 2 = \frac{1}{4^5} + 2$.

$|h(z)| = |4z| = 4|z| = 1$

So $|h(z)| < |f(z)|$ for z on the contour C_2.

ZOZ: is continued

~~3) $f(z) = \frac{z(z-\pi)^2}{(\sin z)^2} = \frac{z(z^2-2\pi z+\pi^2)}{(\sin z)^2} = \frac{z^3-2\pi z^2+\pi^2 z}{(\sin z)^2}$~~

~~determine singularities of $f(z)$.~~

~~a $\sin z = 0 \rightarrow z_0 = k\pi$~~

~~$\sin z_0 = 0 \rightarrow z_0 = k\pi$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$~~

~~b. $z=0$ is a removable singularity.~~

~~$\lim_{z \to 0} f(z) = \lim_{z\to 0} \frac{z^3 - 2\pi z^2 + \pi^2 z}{(\sin z)^2} \overset{L'H}{=} \frac{3z^2 - 4\pi z + \pi^2}{2\sin(z)\cos(z)}$~~

~~at $z=0$:~~

~~$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$~~

Vervolg 5c

According to Rouche's theorem $f(z)$ and $p(z) = f(z) + h(z)$ have the same amount of zero's inside the contour $|z| < \frac{1}{4}$, ~~namely~~

So $p(z)$ has no zeros ~~in the area~~ lying in the area $|z| < \frac{1}{4}$.

In 5b we saw that $p(z)$ has 5 zeros in the area $|z| < 2$, ~~so~~

~~also~~ so $p(z)$ has 5 zeros in the annulus $\frac{1}{4} \leq |z| < 2$